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DIFFUSION APPROXIMATION FOR THERMAL RADIATION IN GASES WITH JUMP BOUNDARY CONDITION

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ABSTRACT

Generalized equations are obtained for radiative diffusion in a non-grey gas and for the energy jump at a wall. By using the expression for the energy jump at a wall, the diffusion approximation for radiation in a gas is improved considerably, the range of validity of the approximation being extended to lower values of optical thickness. AUTHOR

INTRODUCTION

Thermal radiation in a gas is generally complicated by the fact that the radiation passing a given plane originates at points throughout the gas. This leads to the necessity of solving an integral equation [1].[1] However if the gas is optically thick, the mean free path for the radiant energy may be small compared with the overall dimensions of the system. Under these conditions the radiation can be considered as a diffusion process, and the problem reduces to one of solving a modified heat conduction equation [2,3].

Although the diffusion approximation works well in the interior of a gas, it is not accurate near boundaries. Thus, except for extremely large optical thicknesses, considerable error may be made in the calculation

[1]Numbers in brackets designate references at end of paper.

of radiant heat transfer through a gas for given temperatures of the bounding walls. This is illustrated in Fig. 2.

In the present paper the diffusion equations for the thermal radiation in a non-grey gas are first generalized by including spacial derivatives of higher order than the first. The range of applicability of the diffusion approximation is extended by introducing higher order jump boundary conditions at the walls. Shorin [4] has used first order jump boundary conditions, but inasmuch as his results for large optical thickness differ from those obtained from exact solutions, it seems desirable to reexamine the whole problem. Konakov [5] has also considered the effect of walls on radiation. Although he did not use jump boundary conditions, his results are similar to those of Shorin, in that they do not reduce to the correct form for large optical thicknesses [6].

The fact that the temperature in the gas next to a wall should differ from the wall temperature can be seen physically as follows: The radiative flux passing through a plane next to a wall is made up of flux coming from the wall and from gas which, on the average, is a radiation mean free path away from the wall. Thus the average temperature of the radiation passing through the plane next to the wall will lie between the wall temperature and the temperature a mean free path away from the wall. This effect is similar to the temperature jump next to a wall which occurs in heat transfer by conduction in rarefied gases.

Throughout the analysis the heat transfer is assumed to be entirely by radiation, the effects of conduction being assumed negligible. This would be an allowable assumption, for instance, at very high temperatures. When the effects of conduction are not negligible, the conduction and radiation are not strictly additive because of the nonlinearity of the equations. However, calculations made in reference [7] indicate that the error made in assuming that they can be added lies between 0 and 9 percent.

In the next section the equation for the radiant heat transfer in a gas, as well as for the energy jump at a wall will be derived. A non-grey gas bounded by grey walls will be considered. The absorption coefficient as assumed not to vary appreciably over a mean free path for the radiation.

NOMENCLATURE

A	area (see Fig. 1)
c	velocity of light
c_p	specific heat at constant pressure
D	diameter
dE_ν	emission from $d\tau$ at frequency ν which passes through dA
$dE_{\nu 1}$	radiant flux at frequency ν passing through dA from above (see Fig. 1)
$dE_{\nu 2}$	radiant flux at frequency ν passing through dA from below (see Fig. 1)
e_b	total emmisive power of black wall, σT_w^4
e_g	total emissive power of gas, σT^4
e_ν	spectral emissive power of gas given by Planck's distribution function, equation (14)

$e_{\nu b}$ spectral emissive power of black wall given by Planck's distribution function, equation (14)

h Planck's constant, heat transfer coefficient

I defined by equation (20)

k Boltzmann's constant

L thickness of gas space

Q' heat source per unit volume

q radiant heat transfer per unit area

q_ν radiant heat transfer per unit area per unit frequency increment

r radius

T absolute temperature

t time

u gas velocity

x coordinate defined by Fig. 1 measured in direction of gas flow

y coordinate defined by Fig. 1

z coordinate defined by Fig. 1

ϵ wall emissivity

θ,φ angles in spherical coordinate system, φ defined in Fig. 1

κ_r Rosseland mean absorption coefficient defined by equation (18)

κ_s mean absorption coefficient defined by equation (19)

κ_ν spectral absorption coefficient

λ defined by equation (33)

Ω defined by equation (4)

ρ gas density

Γ Gamma function

σ Stefan-Boltzmann constant = 1.714×10^{-9} Btu/(hr)(ft^2)(R^4)

τ volume

ω solid angle, steradians

Subscripts:

c refers to centerline

w refers to wall

z in direction z

0 defined in Fig. 1, also refers to gas at wall

1 refers to inner radius, except in E_1

2 refers to outer radius, except in E_2

BASIC EQUATIONS

To obtain the generalized diffusion equations for the radiation in a gas, together with the jump boundary condition at a wall, consider the radiation streaming through an area element dA at point (x_0,y_0,z_0) from a volume element of gas $d\tau$ at (x,y,z) (Fig. 1). The spectral emissive power e_ν at (x,y,z) can be related to conditions at (x_0,y_0,z_0) by expanding e_ν in a three-dimensional Taylor series about (x_0,y_0,z_0). This gives

$$e_\nu = \sum_{n=1}^{\infty} \frac{1}{n!} \left[(z - z_0)\left(\frac{\partial}{\partial z}\right)_0 + (y - y_0)\left(\frac{\partial}{\partial y}\right)_0 + (x - x_0)\left(\frac{\partial}{\partial x}\right)_0\right]^n e_\nu$$

If we apply the binomial theorem twice to the factor in brackets we obtain

$$e_\nu = \sum_{n=0}^{\infty}\sum_{v=0}^{n}\sum_{s=0}^{v} \frac{(z - z_0)^{n-v}(y - y_0)^{v-s}(x - x_0)^s}{(n - v)!(v - s)!s!} \left(\frac{\partial^n e_\nu}{\partial z^{n-v}\partial y^{v-s}\partial x^s}\right)_0 \tag{1}$$

The symbols are defined in the NOMENCLATURE. The emission from $d\tau$ at frequency ν which passes through dA is

$$dE_\nu = 4\kappa_\nu e_\nu \, d\tau \, \frac{d\omega}{4\pi} \, e^{-\kappa_\nu r} \tag{2}$$

In order for the exponential factor in equation (2) to be applicable, the assumption must be made that the spectral absorption coefficient κ_ν is effectively uniform over a mean free path for the radiation.

The solid angle $d\omega$ equals $dA \cos \theta/r^2$ and

$$d\tau = r^2 \sin \theta \, dr \, d\varphi \, d\theta$$

We can write equations (1) and (2) in spherical coordinates with origin at dA by setting

$$x - x_0 = r \sin \theta \cos \varphi, \; y - y_0 = r \sin \theta \sin \varphi, \; z - z_0 = r \cos \theta$$

From the preceeding equations, the total radiant flux per unit frequency increment passing through dA from above is

$$dE_{\nu 1} = \frac{\kappa_\nu \, dA}{\pi} \sum_{n=0}^{\infty} \sum_{v=0}^{n} \sum_{s=0}^{v} \frac{1}{(n - v)!(v - s)!s!} \left(\frac{\partial^n e_\nu}{\partial z^{n-v} \partial y^{v-s} \partial x^s} \right)_0$$

$$\cdot \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} (r \cos \theta)^{n-v} (r \sin \theta \sin \varphi)^{v-s} (r \sin \theta \cos \varphi)^s$$

$$\cdot \cos \theta \sin \theta \cdot e^{-\kappa_\nu r} \, dr \, d\varphi \, d\theta$$

$$= \frac{dA}{4\pi} \sum_{n=0}^{\infty} \sum_{v=0}^{n} \sum_{s=0}^{v} \Omega(n,v,s) \, \frac{1}{\kappa_\nu^n} \left(\frac{\partial^n e_\nu}{\partial z^{n-v} \partial y^{v-s} \partial x^s} \right)_0 \tag{3}$$

where

$$\Omega(n,v,s) = \frac{\left[1 + (-1)^{v-s}\right] \left[1 + (-1)^s\right] n! \Gamma\left(\frac{n - v + 2}{2}\right) \Gamma\left(\frac{v - s + 1}{2}\right) \Gamma\left(\frac{s + 1}{2}\right)}{(n - v)!(v - s)!s!\Gamma\left(\frac{n + 4}{2}\right)} \tag{4}$$

and Γ is the Gamma function.

In order to obtain the radiation from below, we let θ go from $\pi/2$ to π, instead of from 0 to $\pi/2$, and take the negative value the result:

$$dE_{\nu 2} = \frac{dA}{4\pi} \sum_{n=0}^{\infty} \sum_{v=0}^{n} \sum_{s=0}^{v} (-1)^{n-v} \Omega(n,v,s) \frac{1}{\kappa_\nu^n} \left(\frac{\partial^n e_\nu}{\partial z^{n-v} \partial y^{v-s} \partial x^s} \right)_0 \tag{5}$$

The net radiant heat transfer per unit area per unit frequency increment passing through dA in the direction z is

$$q_{\nu z} = \frac{dE_{\nu 2} - dE_{\nu 1}}{dA}$$

$$= -\frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{v=0}^{n} \sum_{s=0}^{v} \left[1 - (-1)^{n-v}\right] \Omega(n,v,s) \frac{1}{\kappa^n} \cdot \left(\frac{\partial^n e_\nu}{\partial z^{n-v} \partial y^{v-s} \partial x^s} \right)_0 \tag{6}$$

Similar expressions can be obtained for $q_{\nu y}$ and $q_{\nu x}$.

Next the energy jump at a grey wall which is immediately below but not touching the area dA will be obtained. As before, the radiation passing through the area from above is given by equation (3). The energy coming from the wall and passing through dA is made up of radiation emitted by the wall and that reflected from the wall, the latter having been originally emitted by the gas. Thus

$$dE_{\nu 2} = \epsilon e_{\nu b}\, dA + (1 - \epsilon)\, dE_{\nu 1} \tag{7}$$

where ϵ is the wall emissivity and $e_{\nu b}$ is the spectral emissive power for a black wall. Solving equation (7) for $e_{\nu b}$,

$$e_{\nu b} = \frac{1}{\epsilon}\left(\frac{dE_{\nu 2} - dE_{\nu 1}}{dA}\right) + \frac{dE_{\nu 1}}{dA} = \left(\frac{1}{\epsilon} - \frac{1}{2}\right) q_{\nu z} + \frac{1}{2}\, q_{\nu z} + \frac{dE_{\nu 1}}{dA} \tag{8}$$

Substituting equations (3) and (6) in (8) and removing the term for $n=0$ from the summation give

$$e_{\nu b} - e_{\nu 0} = \left(\frac{1}{\epsilon} - \frac{1}{2}\right) q_{\nu z 0} + \frac{1}{8\pi} \sum_{n=1}^{\infty} \sum_{v=0}^{n} \sum_{s=0}^{v} \left[1 + (-1)^{n-v}\right] \Omega(n,v,s) \cdot \frac{1}{\kappa_\nu^n} \left(\frac{\partial^n e_\nu}{\partial z^{n-v} \partial y^{v-s} \partial x^s} \right)_0 \tag{9}$$

where $\Omega(n,v,s)$ is again given by equation (4). Similarly if the wall is above rather than below dA

$$e_{\nu 0} - e_{\nu b} = \left(\frac{1}{\epsilon} - \frac{1}{2}\right) q_{\nu z 0} - \frac{1}{8\pi} \sum_{n=1}^{\infty} \sum_{v=0}^{n} \sum_{s=0}^{v} \left[1 + (-1)^{n-v}\right] \Omega(n,v,s) \cdot \frac{1}{\kappa_\nu^n} \left(\frac{\partial^n e_\nu}{\partial z^{n-v} \partial y^{v-s} \partial x^s}\right)_0 \tag{10}$$

Equations (4), (6), (9), and (10) give the general expressions for the heat flux in the gas and for the energy jumps at the walls. It should be noted that the expression for the heat flux is strictly accurate only for regions at least a radiation mean free path away from the walls, inasmuch as the integration in equation (3) was carried to infinity. However, as is done for heat conduction in rarefied gases, we use the equation throughout the gas and account for the effects of the walls by introducing jump boundary conditions.

In the remainder of the paper we will neglect terms in the series of higher order than the second. (Terms containing zero, first, and second derivatives are retained.) From equations (4), (6), (9), and (10) we then obtain

$$q_{\nu z} = - \frac{4}{3\kappa_\nu} \frac{\partial e_\nu}{\partial z} \tag{11}$$

$$e_{\nu b} - e_{\nu 0} = \left(\frac{1}{\epsilon} - \frac{1}{2}\right) q_{\nu z 0} + \frac{1}{2\kappa_\nu^2}\left(\frac{\partial^2 e_\nu}{\partial z^2}\right)_0 + \frac{1}{4\kappa_\nu^2}\left(\frac{\partial^2 e_\nu}{\partial y^2}\right)_0 + \frac{1}{4\kappa_\nu^2}\left(\frac{\partial^2 e_\nu}{\partial x^2}\right)_0 \tag{12}$$

for a wall below the gas, and

$$e_{\nu 0} - e_{\nu b} = \left(\frac{1}{\epsilon} - \frac{1}{2}\right) q_{\nu z0} - \frac{1}{2\kappa_\nu^2}\left(\frac{\partial^2 e_\nu}{\partial z^2}\right)_0 - \frac{1}{4\kappa_\nu^2}\left(\frac{\partial^2 e_\nu}{\partial y^2}\right)_0 - \frac{1}{4\kappa_\nu^2}\left(\frac{\partial^2 e_\nu}{\partial x^2}\right)_0 \tag{13}$$

for a wall above the gas. The subscripts 0 are omitted in equation (11) because that expression for the heat flux is used throughout the gas. It is correct to second order because the terms containing second derivatives are zero. Equation (11) was obtained by Rosseland [2] and is generally known as the Rosseland approximation. The first term on the right side of equation (12) or (13) was obtained by Shorin [4]. However, his results differ from those obtained here even when the second degree terms are neglected, because he used an expression other than equation (11) for the heat flux close to a wall.

Equations (11) to (13) apply to a single frequency ν. They can be integrated over all frequencies to obtain equations involving the total radiative heat flux and the total emissive power. The spectral emissive power e_ν can be written as a function of ν and the total emissive power $e_g = \sigma T^4$ by writing Planck's distribution function in the form

$$e_\nu = \frac{2\pi h \nu^3}{c^2 \exp\left[h\nu\sigma^{1/4}k^{-1}e_g^{-1/4}\right] - 1} \tag{14}$$

Thus consider e_ν in equations (11) to (13) to be a function of ν and e_g, and apply the rules of partial differentiation of composite functions. If we multiply the equations by $d\nu$ and integrate from 0 to ∞, we obtain

$$q_z = - \frac{4}{3\kappa_r} \frac{\partial e_g}{\partial z} \tag{15}$$

$$e_b - e_{g0} = \left(\frac{1}{\epsilon} - \frac{1}{2}\right) q_{z0} + \frac{1}{2\kappa_s^2}\left(\frac{\partial^2 e_g}{\partial z^2}\right)_0 + \frac{I}{2}\left(\frac{\partial e_g}{\partial z}\right)_0^2 + \frac{1}{4\kappa_s^2}\left(\frac{\partial^2 e_g}{\partial y^2}\right)_0$$

$$+ \frac{I}{4}\left(\frac{\partial e_g}{\partial y}\right)_0^2 + \frac{1}{4\kappa_s^2}\left(\frac{\partial^2 e_g}{\partial x^2}\right)_0 + \frac{I}{4}\left(\frac{\partial e_g}{\partial x}\right)_0^2 \tag{16}$$

for a wall below the gas, and

$$e_{g0} - e_b = \left(\frac{1}{\epsilon} - \frac{1}{2}\right) q_{z0} - \frac{1}{2\kappa_s^2}\left(\frac{\partial^2 e_g}{\partial z^2}\right)_0 - \frac{I}{2}\left(\frac{\partial e_g}{\partial z}\right)_0^2 - \frac{1}{4\kappa_s^2}\left(\frac{\partial^2 e_g}{\partial y^2}\right)_0$$

$$- \frac{I}{4}\left(\frac{\partial e_g}{\partial y}\right)_0^2 - \frac{1}{4\kappa_s^2}\left(\frac{\partial^2 e_g}{\partial x^2}\right)_0 - \frac{I}{4}\left(\frac{\partial e_g}{\partial x}\right)_0^2 \tag{17}$$

for a wall above the gas, where

$$\frac{1}{\kappa_r} = \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial e_\nu}{\partial e_g} \, d\nu \tag{18}$$

$$\frac{1}{\kappa_s^2} = \int_0^\infty \frac{1}{\kappa_\nu^2} \frac{\partial e_\nu}{\partial e_g} \, d\nu \tag{19}$$

$$I = \int_0^\infty \frac{1}{\kappa_\nu^2} \frac{\partial^2 e_\nu}{\partial e_g^2} \tag{20}$$

The derivatives of e_ν with respect to e_g in these equations are obtained from equation (14). For a grey gas, κ_ν is independent of ν and equation (20) becomes

$$I = \frac{1}{\kappa_\nu^2} \frac{\partial^2}{\partial e_g^2} \int_0^\infty e_\nu \, d\nu = \frac{1}{\kappa_\nu^2} \frac{\partial^2 e_g}{\partial e_g^2} = 0 \tag{21}$$

For a non-grey gas it is, of course, necessary to know κ_ν as a function of ν and e_g in order to evaluate equations (18) to (20). The Rosseland mean absorption coefficient κ_r has been calculated for high temperature air in reference 8.

In the following sections equations (15) to (17) will be used for the solution of a few illustrative problems.

ILLUSTRATIVE EXAMPLES AND COMPARISON WITH EXACT SOLUTIONS

Stationary absorbing gas between walls. - Consider first the radiant heat transfer in a stationary gas bounded by two infinite, plane, parallel walls at horizontal planes 1 and 2. For this case q_z is independent of z, the direction normal to the walls, so that equation (15) can be integrated to give

$$e_{g1} - e_g = \frac{3\kappa_r}{4} q_z z \tag{22}$$

or

$$\frac{e_{g1} - e_{g2}}{q_z} = \frac{3\kappa_r L}{4} \tag{23}$$

where L is the distance between the plates and κ_r is assumed constant. From equations (16), (17), and (22) the energy jumps at the walls are

$$\frac{e_{b1} - e_{g1}}{q_z} = \frac{1}{\epsilon_1} - \frac{1}{2} + \frac{9}{32} \kappa_r^2 q_z I \tag{24}$$

and

$$\frac{e_{g2} - e_{b2}}{q_z} = \frac{1}{\epsilon_2} - \frac{1}{2} - \frac{9}{32} \kappa_r^2 q_z I \tag{25}$$

Note that the second and higher order derivatives in equations (16) and (17) are 0 for this case. Adding equations (23) to (25), and taking the reciprocal of the result give

$$\frac{q_z}{\sigma\left(T_{w1}^4 - T_{w2}^4\right)} = \frac{1}{\frac{3\kappa_r L}{4} + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} \tag{26}$$

where e_b has been replaced by its equivalent, σT_w^4.

Equation (26) is plotted against optical thickness $\kappa_r L$ for several values of wall emissivity in Fig. 2. Also plotted for comparison are an exact solution from reference 6, the usual diffusion approximation which neglects the energy jumps at the walls (eq. (23)), and Shorin's result [4].

The improvement of the diffusion approximation by introduction of jump boundary conditions is considerable, the agreement with the exact solution being within 5 percent for all values of $\kappa_r L$. The agreement of the present analysis with the exact solution is also considerably better than that from reference 4, the latter giving values of heat flux about 25 percent too low for large values of $\kappa_r L$. The same comment applies to the results of reference 5.

As the wall emissivity decreases, the radiant heat transfer for a given temperature difference decreases. This is because the heat transfer for a given energy jump at the wall decreases as ϵ decreases (eq. (24)). It is of interest that equation (26) reduces to the correct form for heat transfer between walls with no absorbing gas when $\kappa_r L = 0$.

Consider next the case where the walls are cylindrical and concentric rather than plane. In this case the heat transfer per unit area q is inversely proportional to radius, and equation (15) can be integrated to give

$$e_{g1} - e_g = \frac{3\kappa_r r_1 q_1}{4} \ln \frac{r}{r_1} \tag{27}$$

or

$$\frac{e_{g1} - e_{g2}}{q_1} = \frac{3\kappa_r r_1}{4} \ln \frac{r_2}{r_1} \tag{28}$$

where the subscripts 1 and 2 refer, respectively, to the inner and outer radii. The energy jumps at the inner and outer walls are obtained from equations (16) and (17). The derivatives in those equations are obtained by setting $r = (z^2 + y^2)^{1/2}$ in equation (27), differentiating, and setting $y = 0$. Thus

$$\frac{e_{b1} - e_{g1}}{q_1} = \frac{1}{\epsilon_1} - \frac{1}{2} + \frac{3\kappa_r}{16\kappa_s^2 r_1} + \frac{9}{32}\kappa_r^2 q_1 I \tag{29}$$

and

$$\frac{e_{g2} - e_{b2}}{q_1} = \frac{r_1}{r_2}\left(\frac{1}{\epsilon_2} - \frac{1}{2}\right) - \frac{3\kappa_r r_1}{16\kappa_s^2 r_2^2} - \frac{9}{32}\left(\frac{r_1}{r_2}\right)^2 \kappa_r^2 q_1 I \tag{30}$$

Addition of equations (28) to (30) and use of the relation $r_1 = L/[(r_2/r_1) - 1]$, where L is the radial distance between the walls, give

$$\frac{\sigma\left(T_{w1}^4 - T_{w2}^4\right)}{q_1} = \frac{3\kappa_r L}{4\left[\left(\frac{r_2}{r_1}\right) - 1\right]} \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{\epsilon_1} - \frac{1}{2} + \frac{r_1}{r_2}\left(\frac{1}{\epsilon_2} - \frac{1}{2}\right)$$

$$+ \frac{3}{16}\,\frac{\left(\frac{\kappa_r}{\kappa_s}\right)^2}{\kappa_r L}\,\frac{\left[\left(\frac{r_2}{r_1}\right) - 1\right]^3}{\left(\frac{r_2}{r_1}\right)^2}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right] + \frac{9}{32}\,\frac{q_1 I}{L^2}\,\frac{(\kappa_r L)^2}{\left(\frac{r_2}{r_1}\right)^2}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right] \qquad (31)$$

For small heat flux the last term can be neglected.

A plot of the reciprocal of equation (31) for $\kappa_r/\kappa_s = 1$, small heat flux, and for various values of radius ratio and wall emissivity is given in Fig. 3. For $\kappa_r L = 0$, the correct expression for $q_1/\left[\sigma\left(T_{w1}^4 - T_{w2}^4\right)\right]$ is $1/\{(1/\epsilon_1) + (r_1/r_2)[(1/\epsilon_2) - 1]\}$ if the reflection is diffuse. Thus, equation (31) reduces to the correct expression for $\kappa_r L = 0$ only when $r_1/r_2 = 1$. This is perhaps not surprising because the series in equations (6), (9), and (10) were truncated at terms of order two. For the case of the plane plates the higher-order derivatives were zero, but here they are all non-zero. In order to given an idea of the range of values of $\kappa_r L$ for which the present approximation is valid, curves for the Monte Carlo solution of Perlmutter and Howell [9] are plotted dashed in Fig. 3. It is evident that the approximation is good at rather low values of $\kappa_r L$ when r_2/r_1 is not too large. As r_2/r_1 increases, the range of its applicability moves to higher values of $\kappa_r L$. However, in all cases the improvement over the usual diffusion approximation is considerable. Comparison of these results with the Monte Carlo solution indicates that they could be extrapolated to lower values of $\kappa_r L$ by drawing a smooth curve which is tangent to the curve for the present results and which intersects the known correct ordinate for $\kappa_r L = 0$.

Absorbing gas with heat sources and flow. - Next a simplified version of the radiation to or from a gas containing heat sources and flowing between two parallel walls will be considered. The net rate of heat flow out of a unit volume $\nabla \cdot \vec{q}$ can be equated to the heat source within the volume Q' minus the rate of change of enthalpy within the volume $\rho c_p \, DT/Dt$. From equation (15), the heat flow vector $\vec{q}$ is $-(4/3\kappa_r)\nabla e_g$, if we consider only radiant heat transfer. Thus the energy balance becomes

$$-\frac{4}{3\kappa_r}\nabla \cdot \nabla e_g = -\frac{4}{3\kappa_r}\nabla^2 e_g = Q' - \rho c_p \frac{DT}{Dt} \tag{32}$$

where the properties are assumed constant. Equation (32) will, of course, be the same as the heat conduction equation if we replace $4/3\kappa_r$ by the thermal conductivity and e_g by the temperature. For steady-state conditions and velocities in the x-direction only, the substantial derivative in the last term becomes $u \, dT/dx$. If we neglect heat transfer in the direction of motion, as is usually done for heat conduction in moving fluids, equation (32) becomes

$$\frac{4}{3\kappa_r}\frac{\partial^2 e_g}{\partial z^2} = -Q' + \rho c_p u \frac{\partial T}{\partial x} = \lambda \tag{33}$$

In order to simplify the problem we consider the case where λ is independent of z and the two walls are at the same temperature. Then, if $z = 0$ at the center of the channel, one integration of equation (33) gives

$$\frac{4}{3\kappa_r}\frac{de_g}{dz} = \lambda z \tag{34}$$

Another integration gives

$$e_g - e_{gc} = \frac{3\kappa_r\lambda}{8} z^2 \tag{35}$$

or

$$e_{g0} - e_{gc} = \frac{3\kappa_r\lambda L^2}{32} \tag{36}$$

where the subscript 0 refers to conditions in the gas at the wall, c refers to the center of the channel, and L is the width of the channel. Evaluation of equations (15) and (34), at the wall gives

$$\lambda = \frac{2q_w}{L} \tag{37}$$

so that

$$\frac{e_{g0} - e_{gc}}{q_w} = \frac{3}{16}\kappa_r L \tag{38}$$

The energy jump at the wall is given by equation (16) where the derivatives at the wall are obtained from equations (35) and (37). Thus

$$\frac{e_b - e_{g0}}{q_w} = \frac{1}{\epsilon} - \frac{1}{2} + \frac{3\kappa_r}{4\kappa_s^2 L} + \frac{9}{32} I\kappa_r^2 q_w \tag{39}$$

and

$$\frac{\sigma\left(T_w^4 - T_c^4\right)}{q_w} = \frac{3}{16}\kappa_r L + \frac{1}{\epsilon} - \frac{1}{2} + \frac{\left(\frac{\kappa_r}{\kappa_s}\right)^2}{4\kappa_r L} + \frac{9}{32}\left(\frac{q_w I}{L^2}\right)(\kappa_r\lambda)^2 \tag{40}$$

A plot of the recriprocal of equation (40) is given in Fig. 4 for $\kappa_r/\kappa_s = 1$, small heat flux, and for several values of ϵ. Included for comparison are curves for the Monte Carlo solution of Howell and Perlmutter [10] and the usual diffusion approximation without jump boundary conditions (eq. (38)). The agreement between the present modified diffusion solution and the Monte Carlo solution is reasonably good over the entire range of values of $\kappa_r L$, the agreement, of course, being best for the higher values of $\kappa_r L$. Comparison of these curves with that for the usual diffusion approximation shows, as in the preceding cases, the considerable effect of the energy jump at the wall.

Figure 4 can be used to estimate the radiant heat transfer produced by injecting smoke or particles into the gas stream. The ordinate in Fig. 4 can be written as $h(T_w - T_c)/\left(T_w^4 - T_c^4\right)$, where the heat transfer coefficient $h \equiv q_w/(T_w - T_c)$ is based on the difference between the wall and centerline temperatures. If sufficient smoke is injected into the stream to produce a $\kappa_r L$ of about 2, Fig. 4 indicates that, for temperatures on the order of 4,000° F, heat transfer coefficients on the order of 500 Btu/(hr)(ft^2)(°F) might be obtained. It would not be expected that the introduction of the smoke would greatly affect the pressure drop, so that this appears to be a possible way of obtaining very large heat transfer coefficients in a gas without correspondingly large friction factors.

If we consider a tube, rather than a channel, equation (32) or (33) can be written in cylindrical coordinates as

$$\frac{4}{3\kappa}\frac{1}{r}\frac{d}{dr}\left(r\frac{de_g}{dr}\right) = \lambda \tag{41}$$

Solution of this equation gives

$$e_g - e_{gc} = \frac{3\kappa_r q_w}{4D} r^2 \tag{42}$$

The energy jump at the wall is obtained from equation (17) ($q_{z0} = - q_w$) and the derivatives at the wall in that equation are calculated by setting $r^2 = z^2 + y^2$ in equation (42) and letting y go to zero after differentiating. The final equation is

$$\frac{\sigma\left(T_w^4 - T_c^4\right)}{q_w} = \frac{3\kappa_r D}{16} + \frac{1}{\epsilon} - \frac{1}{2} + \frac{9\left(\frac{\kappa_r}{\kappa_s}\right)^2}{8\kappa_r D} + \frac{9}{32}\frac{q_w I}{D^2} \quad (\kappa_r D)^2 \tag{43}$$

A plot of the reciprocal of equation (43), together with the Monte Carlo solution of reference 9, is plotted in Fig. 5. Good agreement between the two solutions is indicated.

For a gaseous sphere containing heat sources, we can write equation (32) or (33) as

$$\frac{4}{3\kappa}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{de_g}{dr}\right) = \lambda \tag{44}$$

and

$$e_g - e_{gc} = \frac{3\kappa_r q_w}{4D} r^2 \tag{45}$$

For this case we obtain the derivatives in equation (17) by setting $r^2 = z^2 + y^2 + x^2$ in equation (45) and letting y and x go to zero after differentiating. The final equation for the sphere is

$$\frac{\sigma\left(T_w^4 - T_c^4\right)}{q_w} = \frac{3\kappa_r D}{16} + \frac{1}{\epsilon} - \frac{1}{2} + \frac{3\left(\frac{\kappa_r}{\kappa_s}\right)^2}{2\kappa_r D} + \frac{9}{32}\,\frac{q_w I}{D^2}\,(\kappa_r D)^2 \tag{46}$$

Note the similarity between equations (40), (43), and (46).

As a final example consider a radiating gas flowing in an annulus which is heated at the inner radius r_1 and cooled at the outer one r_2. Integration of equation (41) between r_1 and r_2 and use of equations (16) and (17) give, for this case,

$$\frac{\sigma\left(T_{w1}^4 - T_{w2}^4\right)}{q_1} = \frac{3\kappa_r L \ln\left(\frac{r_2}{r_1}\right)}{4\left[\left(\frac{r_2}{r_1}\right) - 1\right]} - \frac{3\kappa_r L}{16}\,\frac{\lambda L}{q_1}\,\frac{\left[\left(\frac{r_2}{r_1}\right)^2 - 1 - 2\ln\left(\frac{r_2}{r_1}\right)\right]}{\left[\left(\frac{r_2}{r_1}\right) - 1\right]^2}$$

$$+ \frac{1}{\epsilon_1} - \frac{1}{2} + \left(\frac{1}{\epsilon_2} - \frac{1}{2}\right)\frac{q_2}{q_1} + \frac{3\left(\frac{\kappa_r}{\kappa_s}\right)^2}{16\kappa_r L}\left[\frac{r_2}{r_1} - 1 + \frac{\lambda L}{2q_1}\right]\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]$$

$$+ \frac{9}{32}\,\frac{q_1 I}{L^2}\,(\kappa_r L)^2\left[1 - \left(\frac{q_2}{q_1}\right)^2\right] \tag{47}$$

and

$$\frac{\lambda L}{q_1} = \frac{2\left[1 - \left(\frac{r_2 q_2}{r_1 q_1}\right)\right]}{\left[\left(\frac{r_2}{r_1}\right) + 1\right]} \tag{48}$$

where L is again the difference between the inner and outer radii, and λ is defined by equation (33). Equations (47) and (48) might, for instance, be used to estimate the heat transfer from an arc of radius r_1 to an abosrbing gas flowing along it through a tube of radius r_2.

CONCLUDING REMARKS

By using second-order jump boundary conditions at walls, the range of validity of the diffusion approximation is extended to comparatively low values of optical thickness. For radiant heat transfer in a stationary gas between flat surfaces, the method gives good results for all values of optical thickness. For concentric cylindrical surfaces good results are obtained for moderate and high values of optical thickness, but the method breaks down before an optical thickness of zero is reached. The results improve as the radius ratio gets closer to 1.

For heat transfer from parallel plates or from a tube to a moving gas with heat sources, the method gives reasonable results for all values of optical thickness. The results for this case indicate that very high heat transfer coefficients might be obtained by introducing smoke or particles into a gas flowing in a passage at high temperatures. The effect of wall emissivity, which is neglected in the usual diffusion approximation, is accounted for in the present method. The results can be applied to a non-grey gas if the spectral absorption coefficient of the gas is known as a function of frequency and temperature.

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FIGURE TITLES

Figure 1. - Sketch for deriving diffusion equations.

Figure 2. - Thermal radiation between flat surfaces with intervening stationary gas. Solid curves calculated from equation (26).

Figure 3. - Thermal radiation between concentric cylindrical surfaces with intervening stationary gas. Solid curves calculated from equation (31) for small heat flux and $\kappa_r/\kappa_s = 1$.

Figure 4. - Thermal radiation from channel walls to flowing gas with heat sources. Solid curves calculated from equation (40) for small heat flux and $\kappa_r/\kappa_s = 1$.

Figure 5. - Thermal radiation from tube to flowing gas with heat sources. Solid curves calculated from equation (43) for small heat flux and $\kappa_r/\kappa_s = 1$.

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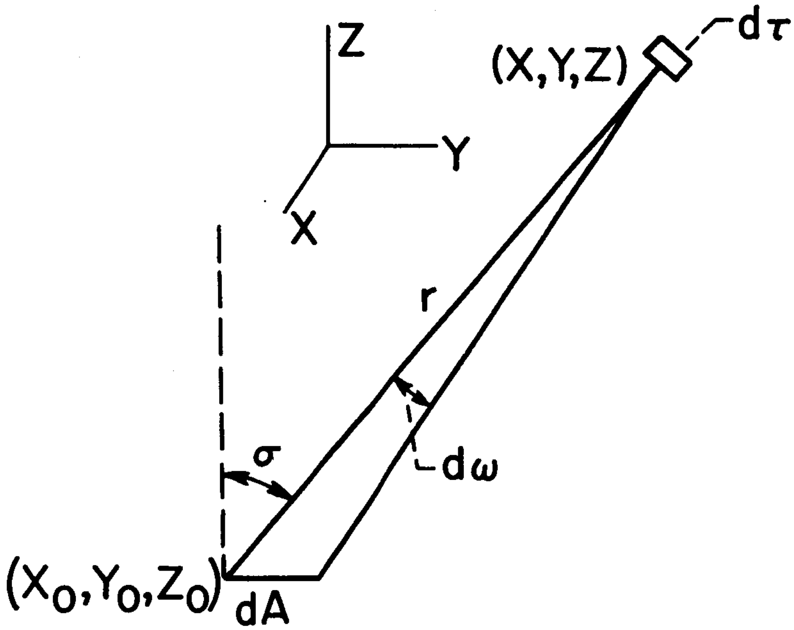

Fig. 1. Sketch for deriving diffusion equations.

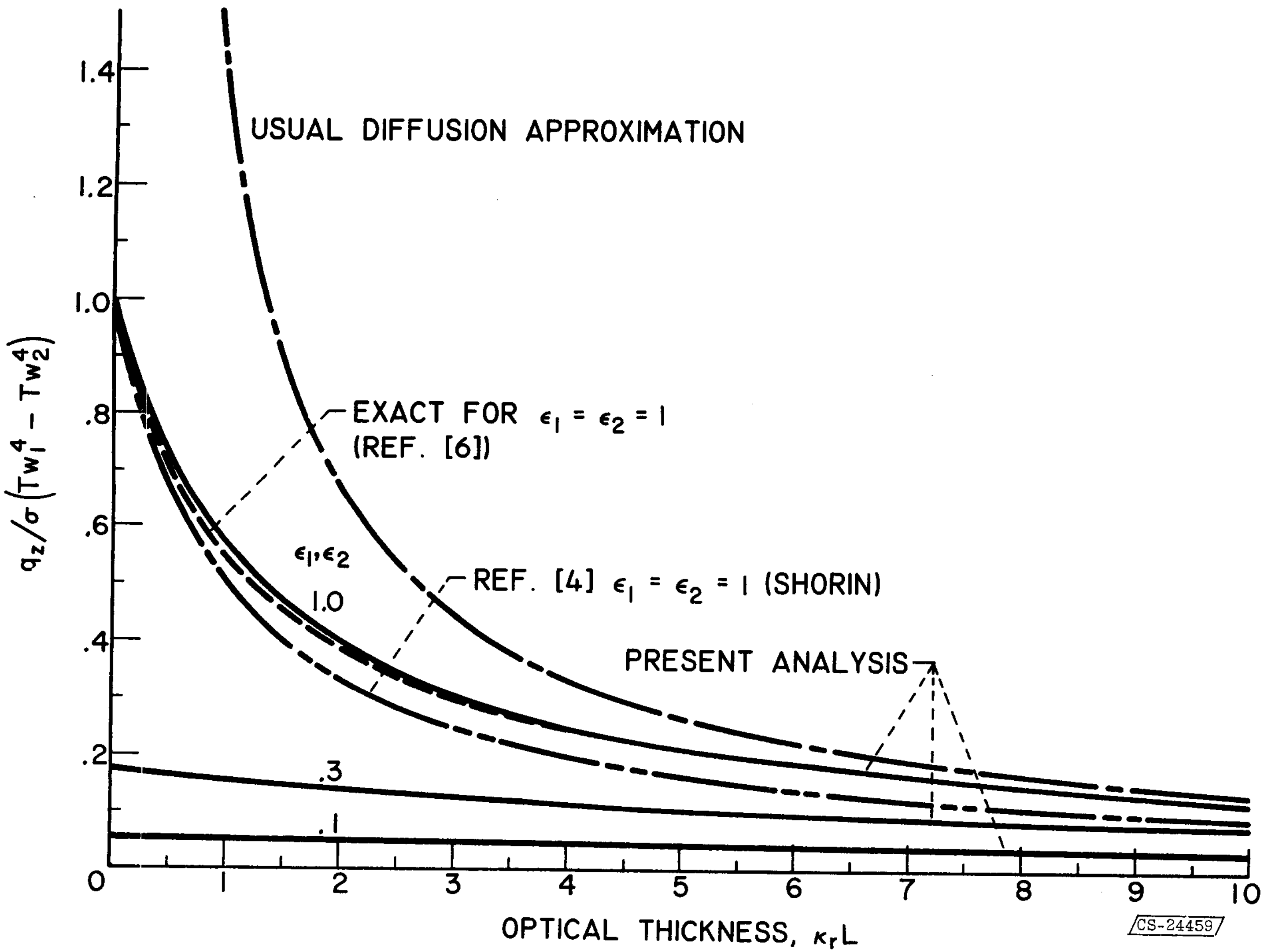

Fig. 2. - Thermal radiation between flat surfaces with intervening stationary gas. Solid curves calculated from equation (26).

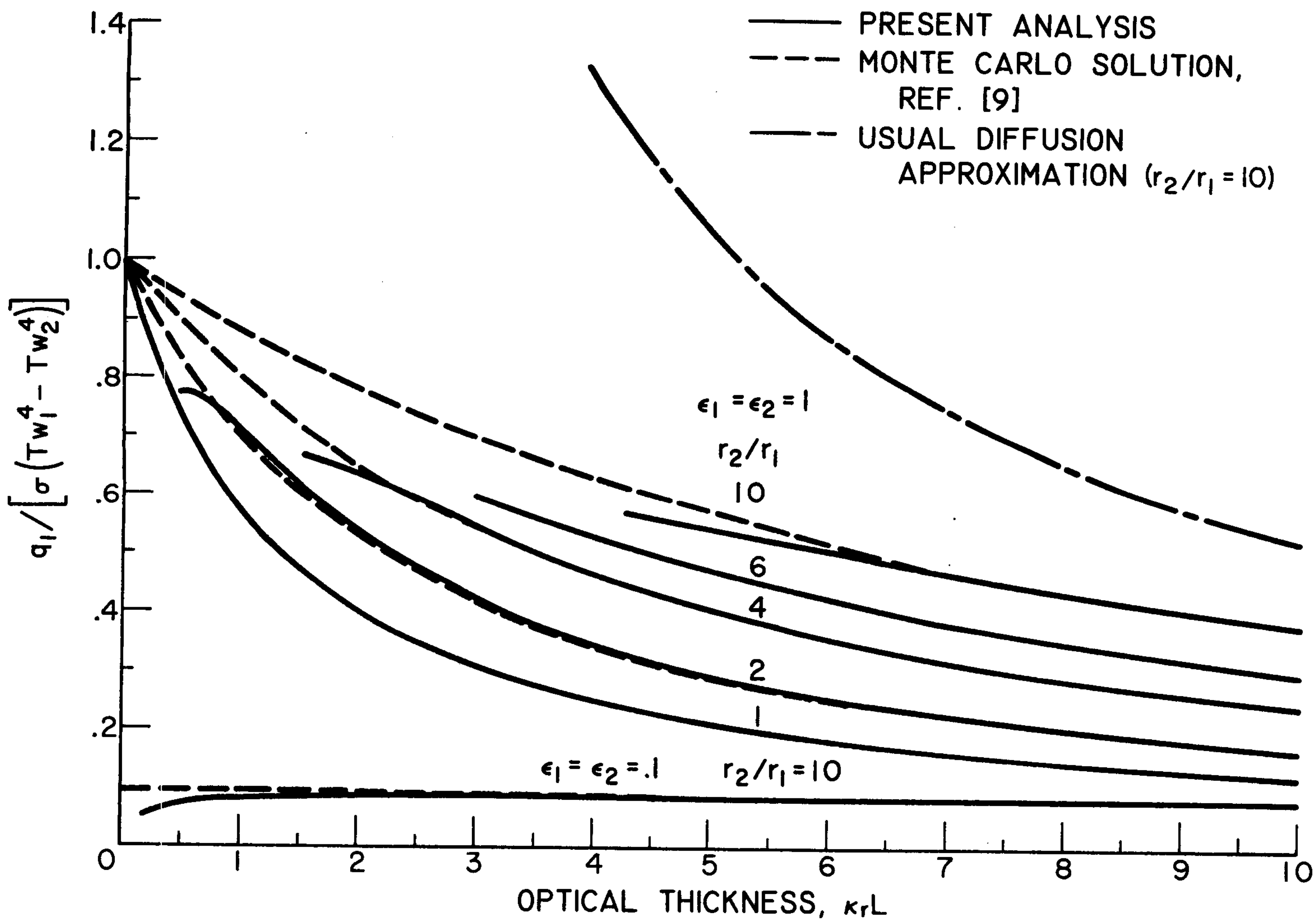

Fig. 3. - Thermal radiation between concentric cylindrical surfaces with intervening stationary gas. Solid curves calculated from equation (31) for small heat flux and $\kappa_r/\kappa_s = 1$.

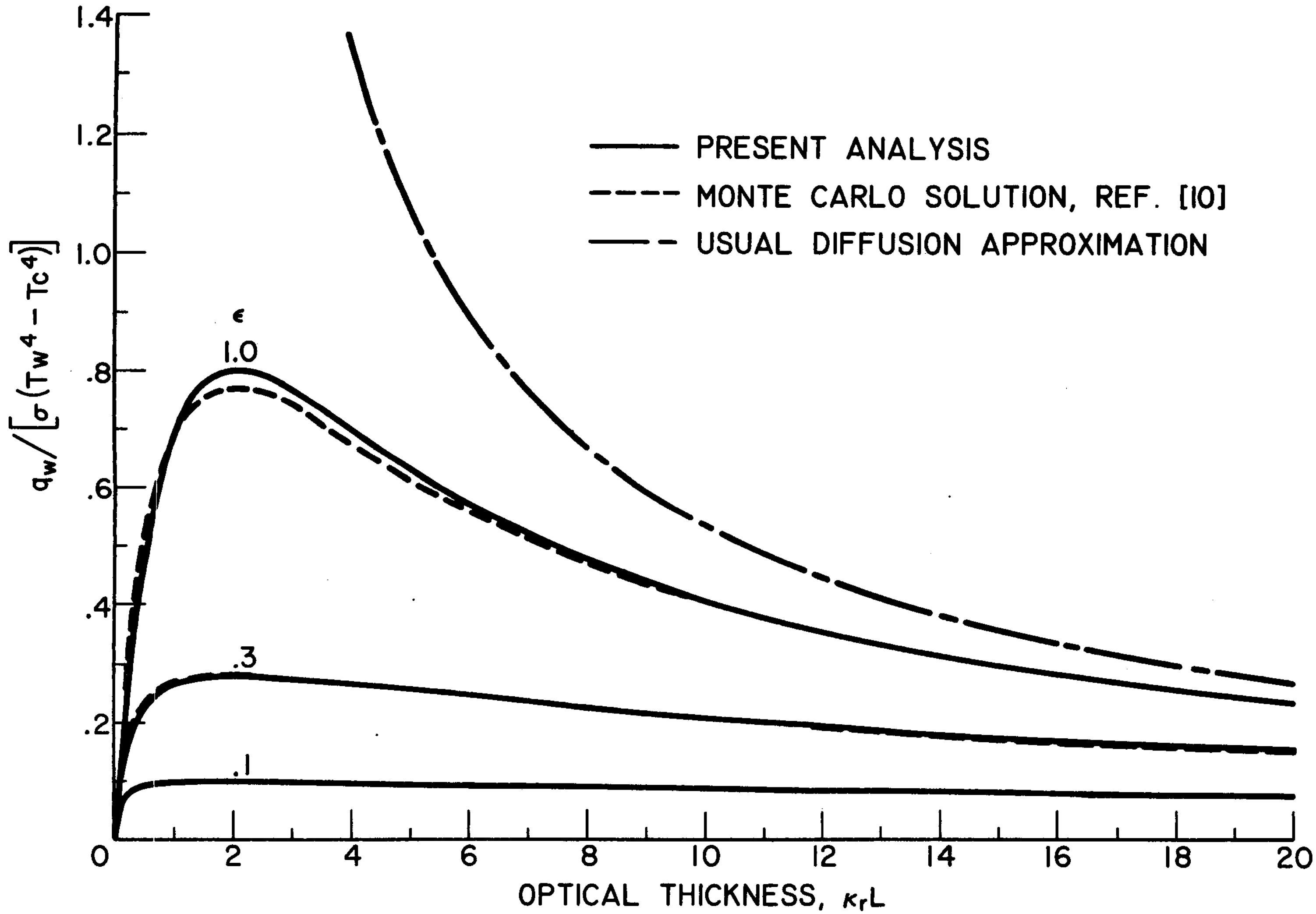

Fig. 4. - Thermal radiation from channel walls to flowing gas with heat sources. Solid curves calculated from equation (40) for small heat flux and $\kappa_r/\kappa_s = 1$.

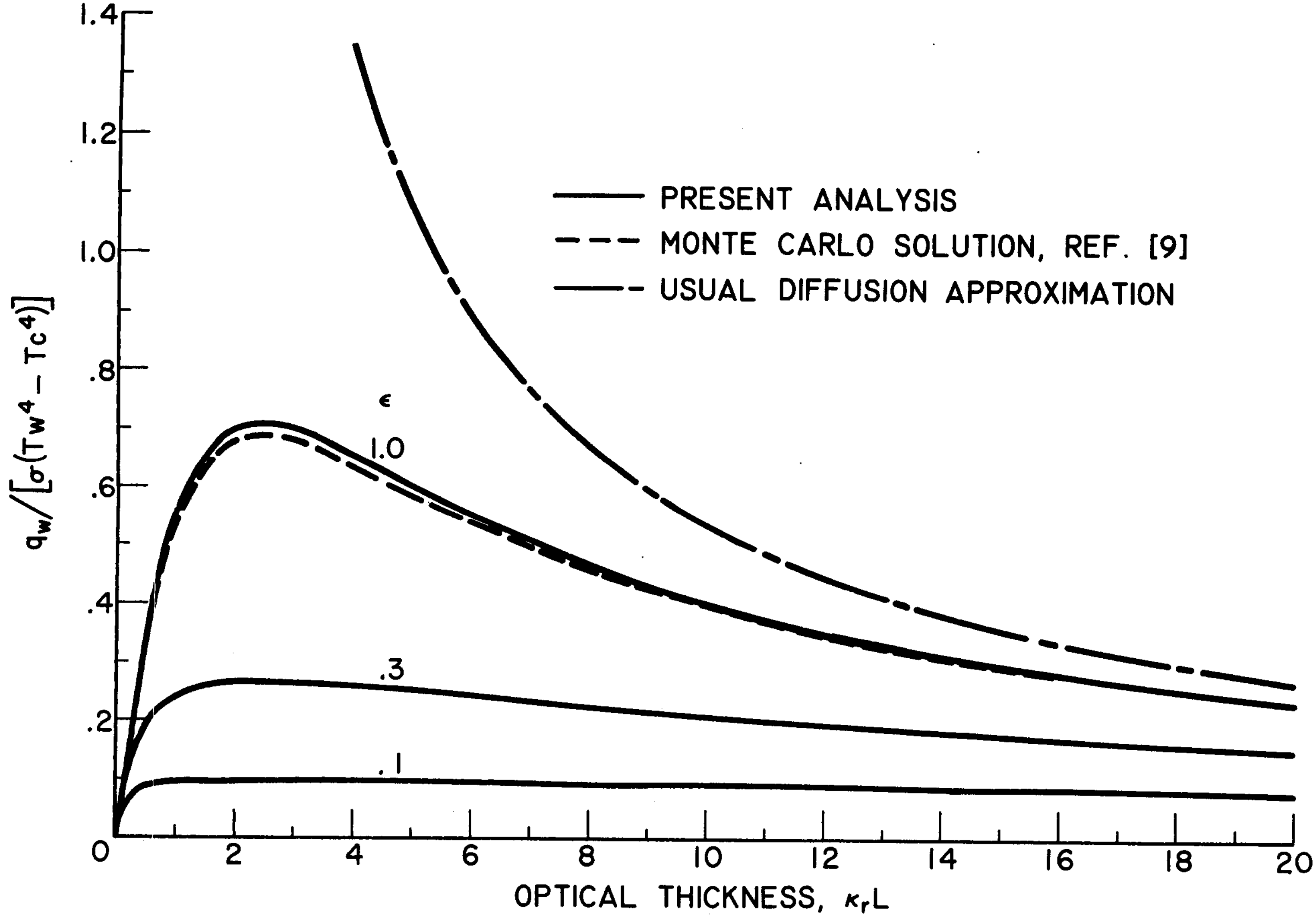

Fig. 5. - Thermal radiation from tube to flowing gas with heat sources. Solid curves calculated from equation (43) for small heat flux and $\kappa_r/\kappa_s = 1$.